

Analysis and Applications of the Capacitive Transformer

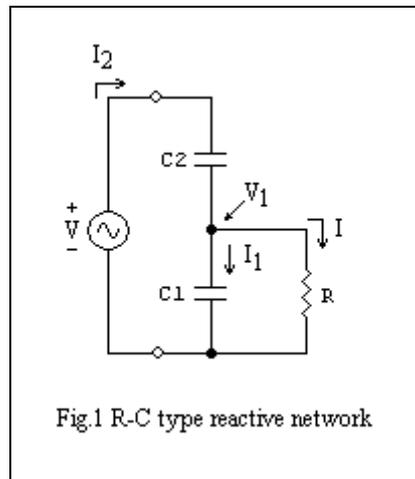
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Wide-band impedance-transformation techniques make common use of magnetic-coupling devices such as transformers (or autotransformers) and baluns for energy transfer between sections of a circuit. Engineering solutions in this area have been well documented in the past and are readily available to the circuit designer from dedicated technical sources.

When operation in a narrow band of frequencies is desired, the engineer usually resorts to a number of passive reactive networks exhibiting impedance-transformation properties. One example is the capacitive transformer, which has received extensive use since the very early days of radio. Surprisingly, however, technical information regarding its operation principle and application data is scarce in the currently available literature. Hence, design work involving this specific network alternatively uses time-consuming trial-and-error methods.

It is the purpose of this article to present a comprehensive analysis of the capacitive transformer and to illustrate how derived results can be incorporated into the design of a simple RF matching network and a grounded-base AF Colpitts oscillator.

Our study begins by considering the reactive network shown in Fig.1. A sinusoidal voltage of amplitude V and radian frequency ω is applied to its input terminals. V_1 is the phasor or complex quantity describing the voltage drop across the load resistor R .



The following equations hold for the network:

$$I_2 = (V - V_1)j\omega C_2$$

$$I_1 = V_1 j\omega C_1$$

$$I = \frac{V_1}{R}$$

$$I_2 = I_1 + I$$

Combining the above equations gives:

$$(V - V_1)j\omega C_2 = V_1 j\omega C_1 + \frac{V_1}{R}$$

Then:

$$\begin{aligned} Vj\omega C_2 &= V_1(j\omega C_1 + j\omega C_2) + \frac{V_1}{R} \\ &= V_1 \left[\frac{1}{R} + j\omega(C_1 + C_2) \right] \\ &= V_1 \left[\frac{1 + j\omega(C_1 + C_2)R}{R} \right] \end{aligned}$$

Solving for V_1 yields:

$$V_1 = V \frac{j\omega C_2 R}{1 + j\omega(C_1 + C_2)R} \quad \dots(1)$$

From the set of basic equations for the network and Eq.(1) we obtain:

$$\begin{aligned} I_2 &= \left[V - V \frac{j\omega C_2 R}{1 + j\omega(C_1 + C_2)R} \right] j\omega C_2 \\ &= V \left[\frac{1 + j\omega C_1 R}{1 + j\omega(C_1 + C_2)R} \right] j\omega C_2 \end{aligned}$$

Input admittance is:

$$\begin{aligned} \frac{I_2}{V} &= \left[\frac{1 + j\omega C_1 R}{1 + j\omega(C_1 + C_2)R} \right] j\omega C_2 \\ &= \frac{(1 + j\omega C_1 R)[1 - j\omega(C_1 + C_2)R]}{1 + \omega^2(C_1 + C_2)^2 R^2} j\omega C_2 \\ &= \frac{1 - j\omega(C_1 + C_2)R + j\omega C_1 R + \omega^2 C_1 R^2 (C_1 + C_2)}{1 + \omega^2(C_1 + C_2)^2 R^2} j\omega C_2 \\ &= \frac{1 - j\omega C_2 R + \omega^2(C_1 + C_2)C_1 R^2}{1 + \omega^2(C_1 + C_2)^2 R^2} j\omega C_2 \\ &= \frac{\omega^2 C_2^2 R + j\omega C_2 [1 + \omega^2 C_1 (C_1 + C_2)R^2]}{1 + \omega^2(C_1 + C_2)^2 R^2} \quad \dots(2) \end{aligned}$$

If $\omega^2(C_1+C_2)^2R^2 \gg 1$, then:

$$\frac{I_2}{V} \cong \frac{C_2^2}{(C_1+C_2)^2 R} + \frac{jC_2[1+\omega^2 C_1(C_1+C_2)R^2]}{\omega(C_1+C_2)^2 R^2}$$

If also $\omega^2 C_1(C_1+C_2)R^2 \gg 1$:

$$\frac{I_2}{V} \cong \frac{C_2^2}{(C_1+C_2)^2 R} + j\omega \frac{C_1 C_2}{C_1+C_2} \quad \dots(3)$$

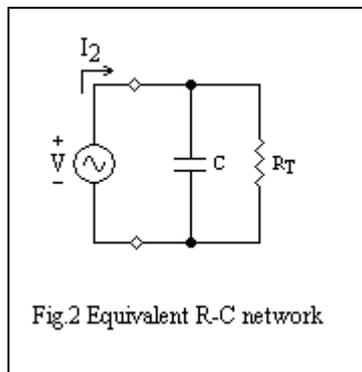
Eq.(3) suggests the equivalent network depicted in Fig.2, as seen from the source side, where:

$$R_T = \left(\frac{C_1+C_2}{C_2} \right)^2 R \quad \dots(3.1)$$

$$= \left(1 + \frac{C_1}{C_2} \right)^2 R$$

and

$$C = \frac{C_1 C_2}{C_1+C_2} \quad \dots(3.2)$$



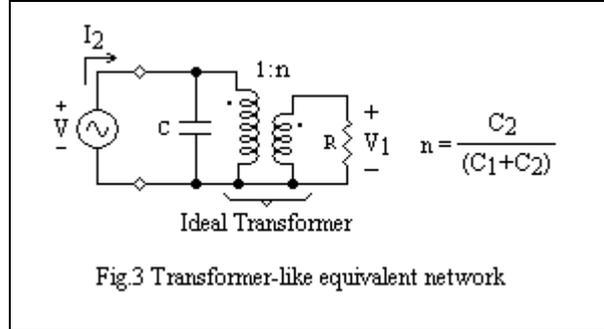
It is interesting to observe that Eq.(1) reduces to:

$$V_1 = V \frac{C_2}{C_1+C_2} \quad \dots(3.3)$$

indicating that the voltage drop across the load resistor is in phase with the applied source voltage and smaller by a fraction $C_2/(C_1+C_2)$. Another equivalent network may be drawn as indicated by Fig.3. Clearly, the R-C network can be made to act as an ideal transformer paralleled by a capacitance C at the primary side, with the resistive load R connected to the secondary circuit and having a 1:n turns-ratio, where:

$$n = \frac{C_2}{C_1 + C_2} \quad \dots(3.4)$$

The original R-C network will be referred to hereafter as the capacitive transformer.



We shall now define the following:

$$G = \frac{1}{R} \quad \dots(4.1)$$

$$Q_E = \omega(C_1 + C_2)R \quad \dots(4.2)$$

$$n^2 G = \frac{1}{R_T} \quad \dots(4.3)$$

$$Q_T = \frac{\omega C}{n^2 G} \quad \dots(4.4)$$

Expressions (4.3) and (4.4) give, respectively, the input conductance and the “Q” at the input port when $Q_E^2 \gg 1$. For this case, the relationship between Q_T and Q_E is:

$$Q_T = Q_E \left(\frac{1-n}{n} \right) \quad \dots(4.5)$$

From Eq.(2), a more general expression for input conductance can be obtained:

$$G_{eq} = \frac{\omega^2 C_2^2 R}{1 + \omega^2 (C_1 + C_2)^2 R^2} \quad \dots(5)$$

Using set of definitions (4), we arrive to an equivalent form for G_{eq} :

$$\begin{aligned} G_{eq} &= \frac{\omega^2 C_2^2 R}{1 + Q_E^2} \\ &= \frac{\omega^2 n^2 (C_1 + C_2)^2 R}{1 + Q_E^2} \end{aligned}$$

$$G_{eq} = \frac{n^2 Q_E^2}{1 + Q_E^2} \cdot \frac{1}{R}$$

$$= n^2 G \cdot \frac{Q_E^2}{1 + Q_E^2} \quad \dots(5.1)$$

Input conductance is then:

$$G_{eq} = n_{eq}^2 G \quad \dots(5.2)$$

where:

$$n_{eq}^2 = n^2 \frac{Q_E^2}{1 + Q_E^2} \quad \dots(5.3)$$

When Q_E^2 is large, the input conductance is approximated by $n^2 G$.

Next, we shall compute the percentage error Δ_1 when $n^2 G$ is used for the input conductance instead of the exact value $n_{eq}^2 G$.

$$\Delta_1 = 100 \times \left(\frac{n^2 G - G_{eq}}{G_{eq}} \right) \%$$

$$= 100 \times \left(\frac{n^2 G - n^2 G \frac{Q_E^2}{1 + Q_E^2}}{n^2 G \frac{Q_E^2}{1 + Q_E^2}} \right) \%$$

$$= \left(\frac{100}{Q_E^2} \right) \%$$

If Q_E is made equal to 10, the percentage error is 1%. So, $G_{eq} = n^2 G$ within 1% when $Q_E = 10$.

Eq.(2) also gives the equivalent input susceptance B_{eq} for the more general case:

$$B_{eq} = \frac{\omega C_2 [1 + \omega^2 C_1 (C_1 + C_2) R^2]}{1 + \omega^2 (C_1 + C_2)^2 R^2} \quad \dots(6)$$

which corresponds to a capacitance:

$$C_{eq} = \frac{C_2 [1 + \omega^2 C_1 (C_1 + C_2) R^2]}{1 + \omega^2 (C_1 + C_2)^2 R^2} \quad \dots(6.1)$$

or equivalently:

$$C_{eq} = \frac{C_2 [1 + (1-n)Q_E^2]}{1 + Q_E^2} \quad \dots(6.2)$$

If Q_E^2 is large, the equivalent input capacitance is $C = C_1 C_2 / (C_1 + C_2)$.

Let's compute now the percentage error Δ_2 when using C for the equivalent input capacitance instead of the exact value C_{eq} .

$$\begin{aligned} \Delta_2 &= 100 \times \left(\frac{C - C_{eq}}{C_{eq}} \right) \% \\ &= 100 \times \left[\frac{\frac{C_1 C_2}{C_1 + C_2} - \frac{C_2 [1 + (1-n)Q_E^2]}{1 + Q_E^2}}{\frac{C_2 [1 + (1-n)Q_E^2]}{1 + Q_E^2}} \right] \% \\ &= 100 \times \left[\frac{(1-n)(1 + Q_E^2) - [1 + (1-n)Q_E^2]}{1 + (1-n)Q_E^2} \right] \% \\ &= \left(-\frac{100n}{1 + (1-n)Q_E^2} \right) \% \end{aligned}$$

In most practical cases it reduces to:

$$\Delta_2 = \left(-\frac{n}{1-n} \right) \%$$

when $Q_E = 10$.

Two examples will follow showing practical application for the derived formulae.

Design of an RF matching network

It is desired to transform a 50-ohm resistive load into a 5k-ohm value at a frequency of 10MHz using a capacitive transformer. The required values for C_1 and C_2 can be readily calculated as follows.

First, we choose $Q_E = 10$. Then, knowing that $\omega = 2\pi \times 10^7$ Rad/sec and $R = 50$ ohms, from (4.2) we obtain:

$$C_1 + C_2 = \frac{10}{2\pi \times 10^7 \times 50} = 3.18nF$$

Next, from Eq.(3.4) and given data:

$$n = \frac{C_2}{C_1 + C_2} = \sqrt{\frac{50}{5000}} = 0.1$$

Then:

$$C_2 = 0.1 \times 3.18nF = 0.318nF$$

$$C_1 = 3.18nF - 0.318nF = 2.862nF$$

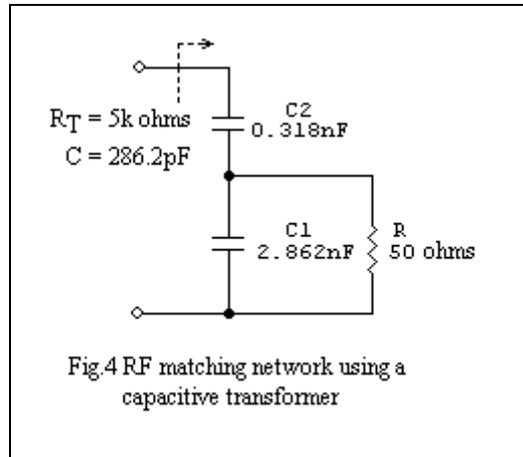
The input capacitance is:

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{2.862 \times 0.318}{3.18} nF = 0.2862nF = 286.2pF$$

with a -0.11% error. The “Q” at the input port is given by Eq.(4.5):

$$Q_T = Q_E \left(\frac{1-n}{n} \right) = 90$$

Fig.4 below depicts the capacitive transformer that will effect the required impedance transformation.



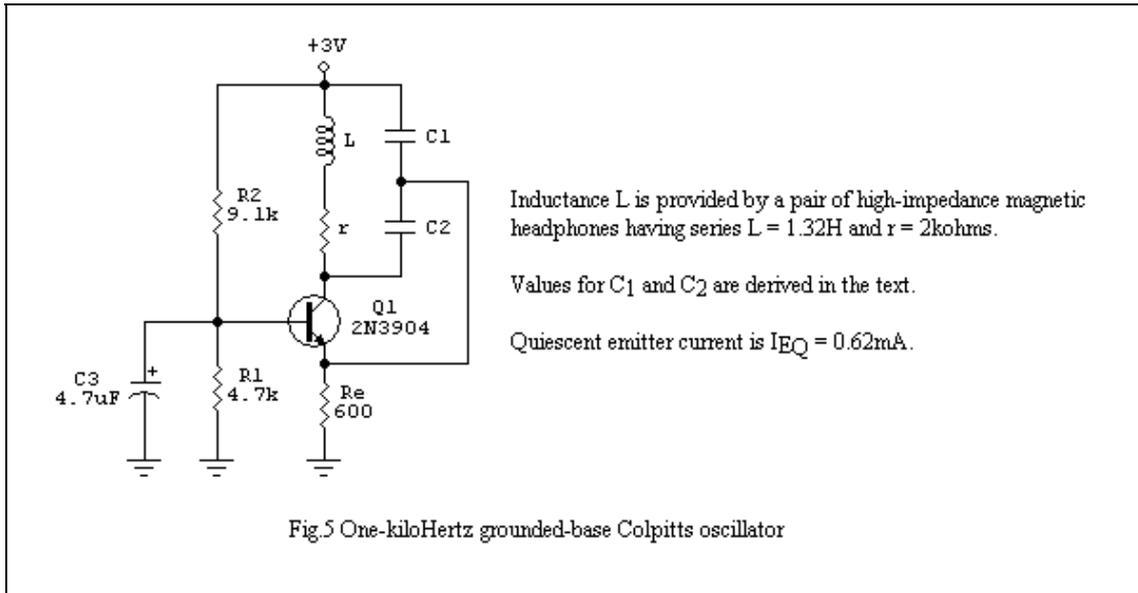
The capacitance $C = 286.2pF$ can be tuned out at 10MHz by an inductance $L = 0.885uH$.

Design of a grounded-base AF Colpitts oscillator

Sine-wave oscillator configurations basically differ from each other in the way signal voltages (or currents) are fed back from the output port to the input, while loop gain is maintained slightly above unity. The Colpitts oscillator, for example, operates successfully thanks to the correct capacitive tapping of the signal voltage existing

across the L-C tuning tank. It constitutes an excellent example of application of the capacitive transformer in an oscillator circuit.

Fig.5 below shows the schematic diagram of a 1kHz grounded-base Colpitts oscillator that was chosen for a proof-of-concept implementation. The inductance L of the tank circuit was provided by a pair of 2k ohms DC-resistance high-impedance magnetic headphones, which doubled as a hearing device. The headphones as measured on a B&K Precision 875A LCR Meter gave $L = 1.32H$ (series inductance) and $r = 2k$ ohms (series resistance), at 1kHz. The power supply was selected to be 3V DC and the quiescent emitter current was set to 0.62mA, for best collector-voltage swing.



Following, loop-gain calculations will be made for the oscillator, identifying capacitive transformer-like operation. C_1 and C_2 will then be computed with the help of previously derived formulae.

The small-signal equivalent model for loop-gain calculation is shown in Fig.6, where:

$$g_m = \frac{I_{E_Q} (mA)}{25} = \frac{1}{h_{ib}} = \frac{1}{40.32} = 0.0248 mhos$$

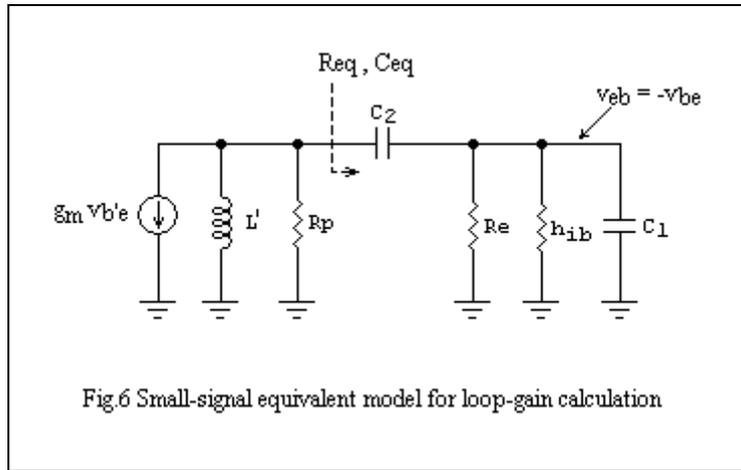
and:

$$L' = L \left(1 + \frac{1}{Q_S^2} \right) = 1.39H$$

$$R_p = r(1 + Q_S^2) = 36.39kohms$$

$$Q_S = \frac{\omega_{osc} L}{r} = \frac{2\pi \times 10^3 \times 1.32}{2000} = 4.147$$

L' and R_p are calculated using the known series-to-parallel impedance transformation pair. If we refer to the parallel combination of R_e and h_{ib} as R , it is not difficult to identify a capacitive transformer-like action due to C_2 , R and C_1 .



Loop gain at the oscillation frequency is given by (please refer to Appendix for proof):

$$\frac{v_{be}}{v_{b'e}} = g_m (R_p // R_{eq}) \cdot n \frac{Q_E^2}{1 + Q_E^2}$$

using capacitive-transformer terminology. If $R_{eq} = 1/G_{eq} \gg R_p$, then:

$$n \geq \frac{1}{R_p g_m} \cdot \frac{1 + Q_E^2}{Q_E^2} = n_{\min}$$

for oscillations to be sustained. The capacitive transformer comprising C_1 , C_2 , R_e and h_{ib} imposes an equivalent capacitance:

$$C_{eq} = \frac{C_2 [1 + (1-n)Q_E^2]}{1 + Q_E^2}$$

across L' . The oscillation frequency $\omega_{osc} = 2\pi f_{osc}$ is given in the Appendix as:

$$\omega_{osc}^2 = \frac{1}{L' C_{eq}} \left(1 + \frac{1}{Q_E Q_{IN}} \right)$$

where:

$$Q_{IN} = \frac{R_p}{\omega_{osc} L'}$$

The design procedure for the oscillator simplifies if we select:

$$n = \frac{C_2}{C_1 + C_2} \ll 1$$

which readily makes $C_{eq} \approx C_2$. Thus, we may write:

$$C_1 + C_2 = \frac{C_{eq}}{n}$$

$$Q_E = \frac{\omega_{osc} C_{eq} R}{n}$$

and:

$$C_1 = \frac{(1-n)C_{eq}}{n}$$

The design steps for our grounded-base Colpitts oscillator will be then:

1. Choose Q_E such that $n_{min} \ll 1$. This will make $C_{eq} \approx C_2$.
2. Obtain C_2 from the expression for ω_{osc} .
3. From that for Q_E obtain the value of n . Check that $n > n_{min}$.
4. Compute C_1 .

Applying the above procedure to our 1kHz oscillator:

1. We select $Q_E = 1$ to obtain $n_{min} = 0.0022 \ll 1$.
2. For this value of Q_E , and knowing that $Q_{IN} = \frac{36.39 \times 10^3}{2\pi \times 10^3 \times 1.39} = 4.17$, we get:

$$C_{eq} \approx C_2 = 22.59 nF$$

3. Knowing that $R = R_c / h_{ib} = 37.78$ ohms, we obtain:

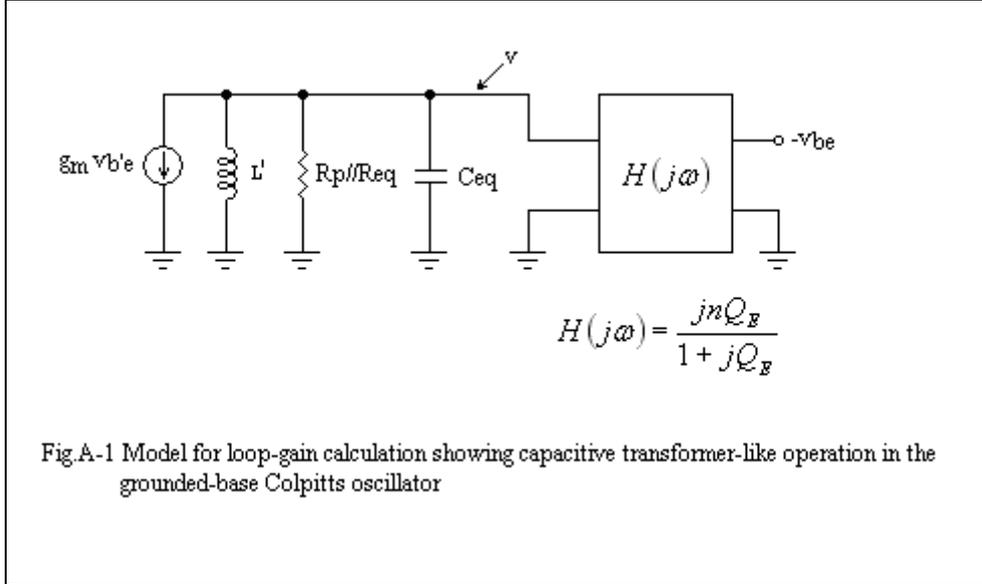
$$n = \frac{\omega_{osc} C_{eq} R}{Q_E} = 0.00536 > n_{min}$$

4. C_1 is computed from $C_1 = \frac{(1-n)C_{eq}}{n} = \frac{1-0.00536}{0.00536} C_{eq} = 185.57 C_{eq} = 4.19 \mu F$

APPENDIX

The grounded-base Colpitts oscillator may be modelled as depicted in the figure below, where $H(j\omega)$ is a functional block equivalent to Eq.(1) from the previous section:

$$H(j\omega) = \frac{V_1}{V} = \frac{jnQ_E}{1 + jQ_E}$$



If $R_{eq} \gg R_P$, loop gain will be expressed by:

$$\begin{aligned} \frac{v_{be}}{v_{b'e}} &= \frac{v}{v_{b'e}} \cdot \frac{v_{be}}{v} = \frac{-g_m j\omega L'}{(1 - \omega^2 L' C_{eq}) + j \frac{\omega L'}{R_P}} \cdot \left(\frac{-jnQ_E}{1 + jQ_E} \right) \\ &= \frac{-g_m \omega L' n Q_E}{1 + jQ_E - \omega^2 L' C_{eq} (1 + jQ_E) + j \frac{\omega L'}{R_P} - \frac{\omega L' Q_E}{R_P}} \\ &= \frac{-g_m \omega L' n Q_E}{\left(1 - \omega^2 L' C_{eq} - \frac{\omega L' Q_E}{R_P} \right) + j \left(Q_E + \frac{\omega L'}{R_P} - \omega^2 L' C_{eq} Q_E \right)} \end{aligned}$$

For oscillations to occur:

$$\frac{v_{be}}{v_{b'e}} = 1 + j0$$

Then:

$$Q_E + \frac{\omega L'}{R_P} - \omega^2 L' C_{eq} Q_E = 0$$

or:

$$\omega_{osc}^2 L' C_{eq} = 1 + \frac{1}{Q_E Q_{IN}}$$

where $Q_{IN} = \frac{R_P}{\omega_{osc} L'}$. In a much more familiar form:

$$\omega_{osc}^2 = \frac{1}{L' C_{eq}} \left(1 + \frac{1}{Q_E Q_{IN}} \right) \quad \dots(A-1)$$

which gives the frequency of the oscillation $\omega_{osc} = 2\pi f_{osc}$.

Loop gain remains as:

$$\begin{aligned} \frac{v_{be}}{v_{b'e}} &= \frac{g_m \omega_{osc} L' n Q_E}{\omega_{osc}^2 L' C_{eq} - 1 + \frac{\omega_{osc} L' Q_E}{R_P}} \\ &= \frac{g_m \omega_{osc} L' n Q_E}{\frac{1}{Q_E Q_{IN}} + \frac{Q_E}{Q_{IN}}} \\ &= \frac{g_m n R_P Q_E^2}{1 + Q_E^2} \end{aligned}$$

For sustained oscillations:

$$\frac{g_m n R_P Q_E^2}{1 + Q_E^2} \geq 1$$

Then:

$$n \geq \frac{1 + Q_E^2}{Q_E^2} \cdot \frac{1}{g_m R_P} = n_{\min} \quad \dots(A-2)$$

The condition $R_{eq} \gg R_p$ implies:

$$R_{eq} = \frac{R}{n_{eq}^2} = \frac{R(1+Q_E^2)}{n^2 Q_E^2} \gg R_p \quad \dots(A-3)$$

or:

$$n^2 \ll \frac{1+Q_E^2}{Q_E^2} \cdot \frac{R}{R_p} \quad \dots(A-4)$$

which imposes an upper limit on n .

In the more general case, the condition for oscillation may be shown to be:

$$ng_m \left(R_p // R_{eq} \right) \frac{Q_E^2}{1+Q_E^2} \geq 1 \quad \dots(A-5)$$

which requires solving a second-degree algebraic equation for n , due to the relationship existing between R_{eq} and this variable, as stated by Eq. (A-3) above.

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